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The coupling of conduction with forced convection in a plane duct

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Abstract—In this paper an analytical solution of the energy equation for laminar convection problems in plane ducts, taking into account the coupling with wall conduction, is presented. The solution is obtained by means of an asymptotic representation of the temperature Laplace transform that enables one to apply the stationary phase method. This entrance solution holds for values of the axial abscissa so high as to permit the calculation of the temperature by means of a few terms of the usual expansion in eigenfunctions. The accuracy of the results, for any coupling parameter, is proved by comparison with those obtained by using an expansion in terms of 120 eigenfunctions.

1. INTRODUCTION

FORCED CONVECTION heat transfer in ducts is important in a large variety of engineering applications, such as the design of compact heat exchangers, nuclear reactors and solar collectors, and a wide literature exists on this subject, both for conventional boundary conditions at the wall–fluid interface, when the temperature or heat flux or a linear combination of temperature and heat flux are given, and for conjugated boundary conditions when the problem is formulated for the entire solid body–fluid medium system and the boundary conditions require continuity of temperature and heat flux at the fluid–solid interface.

Shah and London [1] presented an accurate review of work up to 1976. Luikov *et al.* [2] give the exact solution—reducing the problem, by the generalized Fourier sine transformation, to a singular integral equation for the unknown temperature at the fluid–solid interface. The solution is presented in terms of complicated functions involving definite integrals and series. This solution, without numerical results, is too complex to permit a comparison with the conventional problem.

Davis and Gill [3] examined the effects of axial conduction in the wall on heat transfer with Poiseuille–Couette flow between parallel plates by solving the energy equation in the fluid using the usual eigenfunctions method.

Mori *et al.* [4, 5] investigated, also using the eigenfunctions technique, the effects on the Nusselt number of the boundary conditions (constant temperature or constant heat flux) at the outer wall.

Many authors employed the eigenfunctions method to study the finite wall thermal resistance case, neglecting the solid axial conduction. Hsu [6] tabulated the first ten eigenvalues and gave implicit asymptotic formulae to determine higher eigenvalues; the next 110 values were calculated by Shah and London. Hickman [7] presented asymptotic solutions obtained by expansions based on the properties of the Laplace transform of the temperature

The literature includes several interesting papers following the Shah and London review [1]. For example, Lee and Ju [8] considered the conjugated heat transfer problem of a high Prandtl number fluid.

The effect of axial conduction in the fluid phase must be included for low Peclet number flows; the result is that both energy equations—for solid and fluid phases—are elliptic and more difficult to solve. Papoutsakis and Ramkrishna [9] and Ju and Lee [10] gave, for the conjugated problem with axial conduction, a solution in terms of eigenfunctions applying the matching at the solid–fluid interface. Faghri and Sparrow [11] analysed the influence of simultaneous wall and fluid axial conduction using an elliptic finite-difference method. Wijeysundera [12] developed an analytical method to solve the conjugated problem using Duhamel's superposition technique, also giving a simple procedure for the determination of the eigenvalues of the solution.

Barozzi and Pagliarini [13] used, for the conjugated problem, an iterative technique based on Duhamel's superposition technique, solving the energy equation in the solid by a finite element method. They compared their numerical results with the experimental ones obtained in ref. [14].

The analytical methods presented in the literature are essentially based on asymptotic expansions, and in particular on the modified Graetz technique. These methods are very efficient for high values of x but converge very slowly near the inlet of the duct ($x = 0$). In fact in ref. [1], 120 terms were necessary to obtain the solution at a non-dimensional abscissa (referred to hydraulic diameter of the duct) of 10^{-4} times the Peclet number.

We therefore consider useful an analytical simple solution that describes the entrance temperature field in a duct.

NOMENCLATURE

a_k	coefficients of polynomial P_m, equation (14)
A_{mk}	coefficients defined in equation (15)
b	wall thickness
b_k	coefficients of polynomial Q_m, equation (14)
c_k	coefficient of dimensionless wall temperature, equation (20)
c_p	specific heat of fluid
D_h	hydraulic diameter of the duct
D_v	cylinder parabolic function
F	Laplace transform of dimensionless temperature
h	half-height of the duct
H_k	functions defined by equation (17)
I_k	functions defined by equations (24)
M	confluent hypergeometric function
Nu	local Nusselt number, $q_w D_h/\lambda_f(T_w - T_m)$
p	coupling parameter, $\lambda_f b/\lambda_s h$
Pe	Peclet number, $u_{max} h \rho c_p/\lambda_f$
P_m, Q_m	polynomials defined in equation (14)
q_w	heat flux at interface
s	$t^{1/2}$
t	Laplace variable
T_b	outer wall temperature
T_i	fluid temperature at inlet
T_m	bulk temperature
T_w	interface temperature
u_m	mean velocity of fluid
u_{max}	maximum velocity of fluid
x	dimensionless axial coordinate
$\bar{x}$	$(3/32)x$
y	dimensionless normal coordinate.

Greek symbols

α_k, β_k	coefficients defined in equation (13)
Γ	gamma function
ϑ	dimensionless temperature, $(T - T_i)/(T_b - T_i)$
ϑ_w	dimensionless wall temperature
ϑ_m	dimensionless bulk temperature
λ_f, λ_s	fluid and wall thermal conductivities
τ	$t^{-1/3}$
τ_{mk}	sign-changed roots of polynomial Q_m, equation (15).

In this way it is possible to represent analytically the temperature profiles in the entire field by means of expansions that require few terms to give good accuracy.

2. EQUATIONS AND BOUNDARY CONDITIONS

Figure 1 gives a schematic description of the steady two-dimensional conjugate heat transfer problem considered in this work. The velocity at the inlet is assumed to have a fully developed profile. Fluid is assumed to enter the channel with a uniform temperature T_i.

The temperature field is governed by the energy equations for both phases, and the boundary conditions at the interface require that both the temperature and the heat flux be continuous.

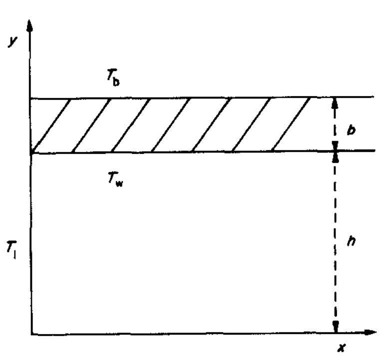

FIG. 1. Schematic diagram of the duct and the coordinate system.

The temperature T_{so} in the solid, neglecting axial conduction, is given by

$$T_{so} = T_w + [T_b - T_w](y - h)/b$$

where T_w is the unknown temperature at the interface and T_b the constant outer wall temperature.

Under the assumption of high Peclet number, the energy equation for the fluid phase may be written in a non-dimensional form as

$$(1 - y^2)\vartheta_x = \vartheta_{yy} \tag{1}$$

where $\vartheta = (T - T_i)/(T_b - T_i)$. The reference lengths for x and y are $h\,Pe$ and h, respectively, where the Peclet number Pe is defined as $u_{max} h \rho c_p/\lambda_f$.

The heat flux continuity condition may be written as

$$\vartheta_w - 1 = -p\vartheta_y(x, 1) \tag{2}$$

where $\vartheta_w = \vartheta(x, 1)$, and

$$p = \lambda_f b/\lambda_s h \tag{3}$$

is the coupling parameter.

The remaining boundary conditions associated with equation (1) are

$$\vartheta(0, y) = \vartheta_y(x, 0) = 0. \tag{4}$$

Equation (1) may be solved by means of the Laplace transform technique. Let $F(t, y)$ be the Laplace transform of ϑ. From equations (1), (2) and (4) one has

$$F_{yy} - t(1 - y^2)F = 0 \tag{5}$$

$$F_y(t,0) = 0; \quad F(t,1) - 1/t = -pF_y(t,1). \tag{6}$$

The solution of this problem is

$$F(t,y) = \exp[(1-y^2)\mathrm{i}s/2]M_1(t,y)F(t,1)/M_1(t,1) \tag{7}$$

where $s = t^{1/2}$, $\mathrm{i} = (-1)^{1/2}$

$$F(t,1) = M_1(t,1)/t \times [(1-\mathrm{i}sp)M_1(t,1) + p(t+\mathrm{i}s)M_2(t,1)] \tag{8}$$

is the Laplace transform of the temperature at the interface and

$$M_1(t,y) = M[(1-\mathrm{i}s)/4, 1/2, \mathrm{i}sy^2];$$

$$M_2(t,y) = M[(5-\mathrm{i}s)/4, 3/2, \mathrm{i}sy^2]. \tag{9}$$

$M(a, b, x)$ is the confluent hypergeometric function.

3. SOLUTION FOR SMALL VALUES OF x

It is not possible to obtain the inverse Laplace transforms of equations (7) and (8) in terms of elementary functions. In order to obtain the solution of the problem which holds for small values of x, we consider the asymptotic behaviour of the confluent hypergeometric function M. Since an integral representation of M is

$$\frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)}M(a,b,z) = \int_0^1 \mathrm{e}^{zr} r^{a-1}(1-r)^{b-a-1}\,\mathrm{d}r = \int_0^1 \exp[zr + (a-1)\log r + (b-a-1)\log(1-r)]\,\mathrm{d}r \tag{10}$$

one has for M_1

$$M_1(t,y) = \frac{\Gamma(1/2)}{\Gamma[(1-\mathrm{i}s)/4]\Gamma[(1+\mathrm{i}s)/4]} I$$

where

$$I = \int_0^1 \exp[\mathrm{i}sy^2 r - ((3+\mathrm{i}s)/4)\log r - ((3-\mathrm{i}s)/4)\log(1-r)]\,\mathrm{d}r. \tag{11}$$

A similar formula holds for M_2.

The solution for small values of x requires an asymptotic representation of M_1 and M_2: such a representation may be obtained by means of the stationary phase method [15]. This technique enables one to evaluate an integral of the form

$$\int_c^d G(r)\,\mathrm{e}^{\mathrm{i}sf(r)}\,\mathrm{d}r$$

as $s \to \infty$ and consists of approximating $G(r)$ and $f(r)$ by their Taylor's formula representation with respect to an initial point which is a stationary point for $f(r)$.

3.1. *Interface temperature and Nusselt number*

In order to obtain an asymptotic representation of the Laplace transform of the temperature at the interface, given by equation (8), we need to evaluate, for high values of s, the integral

$$\int_0^1 G(r)\,\mathrm{e}^{\mathrm{i}sf(r)}\,\mathrm{d}r \tag{12}$$

where $f(r) = r - (1/4)\log r + (1/4)\log(1-r)$ and $G(r) = [r(1-r)]^{-3/4}$ for M_1 and $G(r) = r^{1/4}(1-r)^{-3/4}$ for M_2. As $f'(r) = 0$ for $r = 1/2$, we consider $f(r)$ given by a Taylor expansion of initial point 1/2, obtaining $f = 1/2 - (4/3)(r-1/2)^3 + \ldots$ and we write $G(r)\exp[\mathrm{i}sf(r)]$ as $g(r)\exp[-\mathrm{i}s(4/3)(r-1/2)^3]$, where $g(r) = G(r)\exp\{\mathrm{i}s[f + (4/3)(r-1/2)^3]\}$. Then by expanding $g(r)$ about $r = 1/2$ and by letting $x = (4s/3)^{1/3}(r-1/2)$, we obtain for integral (12) the successive approximations holding for high values of s in terms of I_n, where

$$I_n = \int_{-\infty}^{\infty} x^n\,\mathrm{e}^{-\mathrm{i}x^3}\,\mathrm{d}x.$$

To calculate I_n, the steepest descent method is used; this gives

$$I_n = \tfrac{2}{3}\Gamma\left(\frac{n+1}{3}\right)T_n$$

where $T_n = \cos[(n+1)\pi/6]$ for n even and $T_n = -\mathrm{i}\sin[(n+1)\pi/6]$ for n odd. One therefore has

$$\frac{M_2(t,1)}{M_1(t,1)} = \frac{2}{(1-\mathrm{i}s)}\frac{\sum \alpha_k s^{-k/3}}{\sum \beta_k s^{-k/3}} \tag{13}$$

where the coefficients α_k and β_k are given in Table 1.

From equation (8) one now has for the Laplace transform of the temperature at the interface the following expression:

$$F(t,1) = \frac{\sum \beta_k s^{-k/3}}{t[(1-p\mathrm{i}s)\sum \beta_k s^{-k/3} + 2p\mathrm{i}s\sum \alpha_k s^{-k/3}]}.$$

This equation may be written as

$$F(t,1) = P_m(\tau)/tQ_m(\tau) \tag{14}$$

where $\tau = t^{-1/3}$ and P_m and Q_m are two real polynomials given by

$$P_m = \sum_{k=0}^{m} a_k \tau^k \quad \text{and} \quad Q_m = \sum_{k=0}^{m} b_k \tau^k$$

with $a_0 = 0$, $a_1 = -\beta_0\mathrm{i}/2\alpha_1$, $a_2 = 0$, $a_3 = -\beta_4\mathrm{i}/2\alpha_1$, $a_4 = -\beta_6\mathrm{i}/2\alpha_1$, $b_0 = p$, $b_1 = a_1 + p\alpha_3/\alpha_1$, $b_2 = 0$, $b_3 = a_3 + p\alpha_7/\alpha_1$, $b_4 = a_4 + p\alpha_9/\alpha_1$.

The accuracy of this expansion may be proved by comparing equation (14) with the exact equation (8). Table 2 shows this comparison for several values of p and m.

In order to find the inverse transform of $F(t, 1)$, equation (14) must be rearranged. Let $-\tau_{mk}$ be the

Table 1. Coefficients of the asymptotic expansion of $M_2(1, t)/M_1(1, t)$

k	0	1	2	3	4	5	6	7	8	9
α_k	1.98733	−1.82536i	0	0.198733i	0.058672	0	0.26387	−0.20811i	0	0.048484i
β_k	$2\alpha_0$	0	0	0	$2\alpha_4$	0	$2\alpha_6$	0	0	0

m roots of the polynomial Q_m. Then equation (14) becomes

$$F(t, 1) = (1/t^{4/3})\sum A_{mk}/(\tau+\tau_{mk}). \tag{15}$$

As $1/(\tau+\tau_{mk}) = (\tau^2-\tau_{mk}\tau+\tau_{mk}^2)/(\tau^3+\tau_{mk}^3)$, one has

$$t^{-4/3}/(\tau+\tau_{mk}) = (\tau_{mk}^2 t^{-1/3}-\tau_{mk}t^{-2/3}+t^{-1})/\tau_{mk}^3(\tau_{mk}^{-3}+t). \tag{16}$$

The inverse transform of this function is

$$H_k(x) = 1+\exp(-x/\tau_{mk}^3) \times [3x^{1/3}M(1/3,4/3,x/\tau_{mk}^3)/\tau_{mk}\Gamma(1/3) - (3/2)x^{2/3}M(2/3,5/3,x/\tau_{mk}^3)/\tau_{mk}^2\Gamma(2/3)-1]. \tag{17}$$

Then from equation (15), one has

$$\vartheta_w = \sum_{k=1}^{m} A_{mk}H_k(x). \tag{18}$$

For $p \leqslant 0.01$, this equation, by using the asymptotic representation of M, assumes the simplified form

$$\vartheta_w = \frac{a_1}{b_1}[1-px^{-1/3}/a_1\Gamma(2/3)+p^2x^{-2/3}/a_1^2\Gamma(1/3)]. \tag{19}$$

Table 2. Comparison between the exact value of the Laplace transform of ϑ_w and the value obtained from asymptotic expansion (14)

t	m = 1	3	4	Exact value
		$p = 0.001$		
1	9.9918E−1	9.9910E−1	9.9923E−1	9.9949E−1
10	9.9812E−2	9.9811E−2	9.9814E−2	9.9817E−2
100	9.9585E−3	9.9585E−3	9.9586E−3	9.9586E−3
500	1.9857E−3	1.9857E−3	1.9857E−3	1.9857E−3
1000	9.9100E−4	9.9100E−4	9.9100E−4	9.9100E−4
		$p = 0.1$		
1	9.2434E−1	9.1771E−1	9.2822E−1	9.5176E−1
10	8.4183E−2	8.4109E−2	8.4300E−2	8.4475E−2
100	7.0605E−3	7.0609E−3	7.0638E−3	7.0638E−3
500	1.1635E−3	1.1636E−3	1.1637E−3	1.1637E−3
1000	5.2397E−4	5.2402E−4	5.2405E−4	5.2405E−4
		$p = 1$		
1	5.4990E−1	5.2722E−1	5.6392E−1	6.6361E−1
10	3.4736E−2	3.4609E−2	3.4936E−2	3.5238E−2
100	1.9367E−3	1.9371E−3	1.9392E−3	1.9392E−3
500	2.4420E−4	2.4425E−4	2.4431E−4	2.4431E−4
1000	9.9158E−5	9.9174E−5	9.9186E−5	9.9186E−5
		$p = 10$		
1	1.0887E−1	1.0033E−1	1.1451E−1	1.6477E−1
10	5.0535E−3	5.0267E−3	5.0958E−3	5.1604E−3
100	2.3456E−4	2.3461E−4	2.3492E−4	2.3493E−4
500	2.7434E−5	2.7441E−5	2.7448E−5	2.7448E−5
1000	1.0887E−5	1.0889E−5	1.0891E−5	1.0891E−5

For $p \geqslant 1$, $Q_m(\tau)$, written as $p(1+\varepsilon)$, where $\varepsilon = (1/p)\sum_{k=1}^{m} b_k\tau^k$, may be expanded in a Taylor series, whence equation (14) gives

$$F(t, 1) = P_m\sum(-\varepsilon)^r/pt.$$

The inverse Laplace transform of this function gives the following expression of ϑ_w:

$$\vartheta_w = \frac{1}{p}\sum_{n=0}^{m} c_n\frac{x^{(1+n)/3}}{\Gamma((4+n)/3)} \tag{20}$$

where

$c_0 = a_1$, $c_1 = -a_1b_1/p$, $c_2 = a_1b_1^2/p^2+a_3$,

$c_3 = -a_1(b_3+b_1^3/p^2)/p-a_3b_1/p+a_4$ and

$c_4 = a_1(2b_1b_3/p-b_4+b_1^4/p^3)/p+a_3b_1^2/p^2-a_4b_1/p$.

Equations (18)−(20) can be applied to find the fluid bulk mean temperature and the local Nusselt number. The fluid bulk mean temperature T_m, defined as

$$T_m = \frac{1}{A_c u_m}\int_{A_c} uT\,dA$$

is, in dimensionless form

$$\vartheta_m(x) = (T_m-T_i)/(T_b-T_i) = (3/2)\int_0^1 (1-y^2)\vartheta(x,y)\,dy$$

and, by using equations (1) and (2), may be written as

$$\vartheta_m(x) = (3/2p)\int_0^x [1-\vartheta_w(\bar{x})]\,d\bar{x}.$$

Table 3. Comparison between the Nusselt numbers obtained from equations (18) and (19) and that presented in ref. [1] : $p = 0.01$

$\bar{x}$	Ref. [1]	Equation (19)	Equation (18)
0.0001	27.071	27.081	27.075
0.0002	21.516	21.532	21.517
0.0004	17.185	17.208	17.186
0.0006	15.120	15.147	15.121
0.0008	13.836	13.867	13.837
0.001	12.935	12.969	12.936
0.002	10.616	10.663	10.617
0.004	8.953	9.024	8.955
0.006	8.277	8.370	8.280
0.008	7.939	8.061	7.949
0.01	7.760	7.928	7.792

Table 4. Comparison between the Nusselt numbers obtained from equations (18) and (20) and that presented in ref. [1] : $p = 1$

$\bar{x}$	Ref. [1]	Equation (20)	Equation (18)
0.0001	31.710	31.715	31.715
0.0002	25.202	25.201	25.202
0.0004	20.086	20.085	20.086
0.0006	17.630	17.627	17.630
0.0008	16.096	16.092	16.096
0.001	15.015	15.010	15.015
0.002	12.207	12.193	12.207
0.004	10.139	10.098	10.140
0.006	9.251	9.170	9.252
0.008	8.768	8.635	8.772
0.01	8.481	8.289	8.494

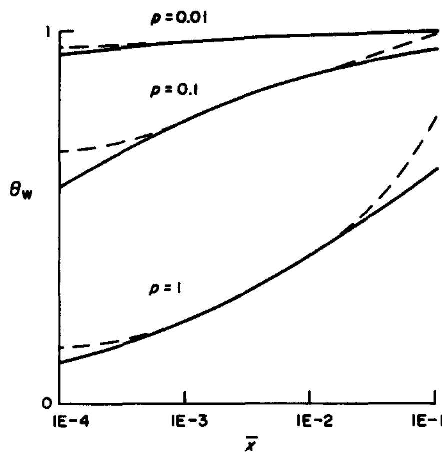

FIG. 2. Comparison between the interface temperature given by equation (18) (solid curves) and that obtained from a four-term expansion in eigenfunctions (dashed curves).

The local Nusselt number $Nu(x) = q_w D_h/\lambda_f(T_w - T_m)$, D_h being the hydraulic diameter of the duct, is given by

$$Nu(x) = 4\vartheta_y(x,1)/[\vartheta_w(x) - \vartheta_m(x)].$$

The good accuracy of the representations of ϑ_w, given by equations (18) and (19) for small values of p and by equations (18) and (20) for large values of p, is shown in Tables 3 and 4, respectively, where the Nusselt number, obtained from equations (18)–(20) is compared with that presented in ref. [1] by using 120 terms of the eigenfunction method. In these tables, the dimensionless axial abscissa $\bar{x}$ is that defined in ref. [1], and $\bar{x} = (3/32)x$.

In order to describe analytically the temperature distribution in the entire field, we compare the interface temperature ϑ_w obtained from equation (18) with the value obtained by using four terms of the expansion in eigenfunctions. Table 5 and Fig. 2 show that equation (18) gives good accuracy at least up to $\bar{x} = 0.01$. Moreover, in the range $0.001 \leqslant \bar{x} \leqslant 0.01$, the two methods give practically the same values.

Thus for $0 \leqslant \bar{x} \leqslant 0.01$, the interface temperature is well described by equation (18); for $\bar{x} \geqslant 0.01$ two eigenfunctions are sufficient to obtain good accuracy.

3.2. *Temperature at the axis of the duct*

From equation (7) and bearing in mind that $M_1(t,0) = 1$, the Laplace transform of the temperature at the axis of the duct ($y = 0$) is given by

$$F(t,0) = \exp[\mathrm{i}s/2]/t[(1-p\mathrm{i}s)M_1(t,1) + p\mathrm{i}s(1-\mathrm{i}s)M_2(t,1)]$$

and therefore, from the asymptotic expansion of $M_1(t, 1)$ and $M_2(t, 1)$, one has

$$F(t,0) = \frac{s^{1/3}\Gamma[(1-\mathrm{i}s)/4]\Gamma[(1+\mathrm{i}s)/4]}{\Gamma(1/2)[(1-p\mathrm{i}s)\sum\beta_k s^{-k/3} + 2p\mathrm{i}s\sum\alpha_k s^{-k/3}]}. \tag{21}$$

The asymptotic representation of the gamma function at the leading term gives

$$\Gamma[(1-\mathrm{i}s)/4]\Gamma[(1+\mathrm{i}s)/4] \simeq 4\pi\,\mathrm{e}^{-\pi s/4}/s^{1/2}$$

and therefore equation (21) may be written as

$$F(t,0) = \frac{2\pi^{1/2}t^{-1/12}\exp(-\pi t^{1/2}/4)}{|\alpha_1|\,t^{4/3}Q_m(\tau)} \tag{22}$$

Table 5. Comparison between ϑ_w obtained from equation (18) and from a four-term expansion in eigenfunctions

	$p = 0.01$		$p = 0.1$		$p = 1$	
$\bar{x}$	Eigenfunction method	Equation (18)	Eigenfunction method	Equation (18)	Eigenfunction method	Equation (18)
1E−4	0.958	0.937	0.673	0.579	0.150	0.113
2E−4	0.960	0.950	0.687	0.638	0.161	0.139
4E−4	0.964	0.960	0.712	0.693	0.180	0.171
6E−4	0.967	0.965	0.731	0.723	0.196	0.192
8E−4	0.969	0.969	0.747	0.743	0.210	0.208
1E—3	0.971	0.971	0.760	0.758	0.222	0.221
2E−3	0.977	0.977	0.801	0.801	0.266	0.266
4E−3	0.982	0.982	0.839	0.838	0.317	0.317
6E−3	0.984	0.984	0.858	0.858	0.350	0.350
8E−3	0.986	0.986	0.871	0.871	0.375	0.375
1E−2	0.987	0.987	0.880	0.880	0.395	0.395

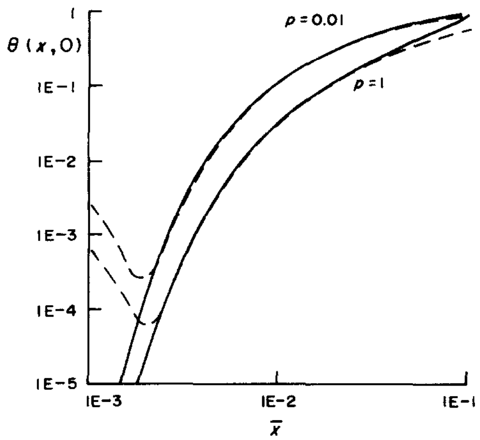

FIG. 3. Comparison between the temperature at the axis of the duct given by equation (23) (solid curves) and that obtained from a four-term expansion in eigenfunctions (dashed curves).

where $Q_m(\tau)$ is the polynomial defined in the previous section.

The inverse transform of equation (22) is

$$\vartheta(x,0) = (2\pi^{1/2}/|\alpha_1|)\sum_{k=1}^{m} A_{mk}[\tau_{mk}^2 I_k(v_1,x) - \tau_{mk} I_k(v_2,x) + I_k(v_3,x)]/\tau_{mk}^3 \quad (23)$$

where

$$I_k(v,x) = 2^{-v}\pi^{-1/2}\,e^{-dx}\int_0^x e^{dx}x^{-v-1/2} \times \exp[-(\pi/4)^2/8x]D_{2v}(\pi/4(2x)^{1/2})\,dx \quad (24)$$

$v_i = (5-4i)/12$, $d = 1/\tau_{mk}^3$ and $D_v(z)$ is the parabolic cylinder function.

Equation (24) may be greatly simplified when $m = 1$ (i.e. neglecting terms of the order of $\tau^3 = t^{-1}$ in the polynomial $Q_m(\tau)$) and $p \leqslant 0.1$.

In this case it is

$$I_k(v,x) = 2^{-v}\pi^{-1/2}d^{-1}x^{-v-1/2} \times \exp[-(\pi/4)^2/8x]D_{2v}(\pi/4(2x)^{1/2}) \quad (25)$$

where $d = b_1^3/p^3$, and if x is sufficiently small one has

$$I_k(v,x) = 2^{-6v}\pi^{2v-1/2}d^{-1}x^{-2v-1/2}\exp[-(\pi/4)^2/4x].$$

In Fig. 3, $\vartheta(x, 0)$ given by equation (23) with $m = 1$ is compared with that obtained from a four-term expansion in eigenfunctions that presents errors less than 0.01% for $\bar{x} \geqslant 0.005$. The present solution practically coincides with the asymptotic one in the range $0.005 \leqslant \bar{x} \leqslant 0.05$ and therefore holds up to values of x for which the asymptotic expansion with a few terms gives accurate results.

4. CONCLUDING REMARKS

The purpose of this work was to represent analytically the temperature distribution in a duct for the laminar convection–wall conduction problem when the velocity profile is fully developed.

As it is possible to describe the temperature profiles by means of two terms of the usual expansion in eigenfunctions with a percentage error of less than 0.01%, starting from $x/(h\,Pe) = 0.1$, where h is the half height of the duct, the entire temperature field is analytically represented if one obtains an entrance solution holding up such a value of the axial abscissa.

This entrance solution has been obtained by means of an asymptotic evaluation of the temperature Laplace transform (i.e. for high values of the Laplace variable) and an application of the stationary phase method.

These results describe the temperature field with good accuracy for any value of the coupling parameter p and are simpler than those obtained by using the expansion in eigenfunctions for small values of the axial abscissa. In fact, for $x/(h\,Pe) = 0.001$, 120 terms of such an expansion are necessary for obtaining good accuracy; more terms are necessary for smaller values of x.

The results refer to values of p in the range 0.01–1, because the values of practical interest of p are in the range 0.001–1. If $p < 0.001$ the values of Nu practically coincide with those obtained from the solution for the case of constant inner wall temperature, whereas if $p \gg 1$ they coincide with those obtained from the solution for the case of constant heat flux at the inner wall.

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COUPLAGE DE LA CONDUCTION AVEC LA CONVECTION FORCEE DANS UN CANAL PLAN

Résumé—On présente une solution analytique de l'équation d'énergie pour le problème de la convection laminaire dans un canal plan en tenant compte du couplage avec la conduction dans la paroi. La solution est obtenue au moyen d'une représentation asymptotique de la transformée de Laplace de la température qui permet d'appliquer la méthode de phase stationnaire. La solution d'entrée est valable pour des valeurs de l'abscisse axiale si élevées qu'il est possible de calculer la température avec peu de termes du développement en fonctions propres. La précision des résultats, pour une valeur quelconque du paramètre de couplage, est prouvée par la comparaison avec les résultats obtenus avec un développement de 120 fonctions propres.

DIE KOPPLUNG VON WÄRMELEITUNG MIT ERZWUNGENER KONVEKTION IN EINEM EBENEN KANAL

Zusammenfassung—In dieser Arbeit wird eine analytische Lösung für die Energiegleichung bei laminarer Konvektion in ebenen Kanälen dargestellt, die die Kopplung mit der Wärmeleitung in der Wand berücksichtigt. Die Lösung wird Hilfe einer asymptotischen Darstellung der Laplace-Transformation der Temperatur erhalten, welche die Anwendung der stationären Phasenmethode ermöglicht. Diese Eintrittslösung gilt für ausreichend hohe Werte der Abszisse, um die Temperaturberechnung mit Hilfe weniger Terme der gewöhnlichen Reihenentwicklungen in Eigenfunktionen zu erlauben. Die Genauigkeit wird für jeden Wert der Kopplungsparameter durch Vergleich mit den Ergebnissen nachgewiesen, welche sich bei einer Entwicklung mit 120 Eigenfunktionen ergeben.

СОПРЯЖЕННЫЙ КОНВЕКТИВНЫЙ ТЕПЛООБМЕН В ПЛОСКОМ КАНАЛЕ

Аннотация—Представлено аналитическое решение уравнения сохранения энергии для ламинарного течения в плоских каналах в постановке сопряженной задачи конвективного теплообмена. Решение получено с помощью асимптотического представления преобразованной по Лапласу температуры, позволяющего применить метод стационарной фазы. Это начальное решение используется для случая, когда значения осевой абсциссы не слишком велики, чтобы рассчитать температуру с помощью нескольких членов обычного разложения по собственным функциям. Точность результатов для любого значения параметра связи доказывается путем сравнения с результатами, полученными при использовании разложения, включающего 120 собственных функций.